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Precision Spacecraft Attitude Estimators Using an Optical Payload Pointing System

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The advancement of satellite program technology mandates that issues affecting future spacecraft system development cost be resolved, while continuing to meet smaller size, lower power, and more stringent mission requirements. In the specific area of the spacecraft attitude determination system, an optical payload pointing system can serve a dual purpose: to support the payload mission and to provide precise attitude information for the spacecraft functions. As a result, star trackers or other precise reference systems can be eliminated to reduce the development cost. Additional benefits include performance robustness to spacecraft motion or disturbance and potential cost saving due to weight reduction, e.g., launch cost. Attitude estimators using an optical payload pointing system and strap-down gyros are derived in detail for two distinct formulations: spacecraft body and inertial formulations. Alternative Kalman filter equation derivations are shown using the skew symmetric matrix properties of the attitude kinematics equation. It is also shown that both formulations produce identical solutions. However, the body formulation is conceptually easier to understand, and the inertial formulation requires less processing time. To show the spacecraft attitude estimator performance using an optical payload pointing system, an example is shown for an agile low-Earth-orbit satellite.

Nomenclature

$\hat{A}$ = attitude knowledge error estimate state in inertial coordinate, rad
$\hat{b}$, $\hat{B}$ = gyro bias estimate in body and inertial coordinate, rad/s
b_{gyro} = gyro bias, rad/s
F_B, F_I = state equation matrices for the body and inertial formulation
H, h = measurement equation
$I_{3\times 3}$ = 3×3 identity matrix
$Q(t)$, $R(t)$ = state and measurement covariance matrices
$\boldsymbol{R}_I^{\text{Star}}$ = star vector in inertial frame
$\boldsymbol{R}_I^{\text{StarCat}}$ = estimated star vector from star catalog
SF = gyro scale factor error, ppm
T_1^2 = direction cosine coordinate transformation from 1 to 2 coordinate
$w(t)$, $v(t)$ = zero mean and Gaussian white noise processes
$x(t)$, $y(t)$ = state and measurement variables
$\hat{\alpha}$ = attitude knowledge estimate error in body coordinate, rad
β_i = inner gimbal angle, rad
β_o = outer gimbal angle, rad
Δt = discrete Kalman filter sampling rate, s
δx = error state vector
$\delta(\tau)$ = delta function
η_a = rate random walk, rad/s$^{3/2}$
η_r = rate noise or angle random walk, rad/s$^{1/2}$
Φ = discrete version of F
$\boldsymbol{\Omega}$ = attitude rate vector, rad/s
ω_{true} = true spacecraft body rate, rad/s
ω_{gyro} = gyro reading, rad/s
$0_{m\times n}$ = $m \times n$ matrix with zero elements

Subscripts and Superscripts

B = spacecraft body frame
$\hat{B}$ = estimated spacecraft body frame
FP = focal plane
I = inertial frame

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Introduction

PRECISE attitude determination systems traditionally utilize fixed-head or gimbal star trackers and body-mounted gyros to estimate the spacecraft orientation in the inertial coordinate frame. The fixed-head star-tracker configuration has worked well for the last decade or so. However, star trackers or other precision sensors are usually expensive, and there is a need to reduce the spacecraft development cost due to budget limitations.[1] For a spacecraft that carries an optical payload pointing system as shown in Fig. 1, the optical payload can serve a dual purpose: 1) to support the payload primary mission and 2) to provide a precise attitude reference for the spacecraft functions. As a result, star trackers or other precise attitude reference sensors can be eliminated. The elimination of this sensor hardware reduces the system development/procurement cost and also reduces the total system weight, which may potentially provide additional cost savings, e.g., launch cost, propellant, etc.

Attitude determination algorithms using fixed-head or gimbal star trackers have been developed and well published.[2–12] These algorithms essentially determine the spacecraft attitude using either extended Kalman filter mechanization or least-squares procedures minimizing a generalized cost function. References 9–11 and the references therein develop least-squares algorithms based on minimizing Wahba's objective function. The primary advantage of this method is that an a priori estimate of the attitude is not required. In contrast, the Kalman filter algorithm requires that a nonlinear estimation problem be linearized about an a priori estimate of the attitude. However, the Kalman filter approach has an advantage that error states other than the attitude can be estimated by simply adding them to the state vector. This paper elaborates on the Kalman filter method for spacecraft attitude estimation further and derives two formulations of the filter state and measurement equations. Additional references on the subject of Kalman filtering applications for spacecraft attitude estimation can be found in the classic tutorial paper of Lefferts et al.[2]

Standard Kalman filter mechanizations for attitude determination systems using star trackers typically include three attitude and three gyro bias errors as the filter states. Kalman filter equations for the six-state filter have been previously derived for two distinct formulations: body and inertial coordinate formulations. References 2 and 3 show the body formulation case for fixed-head star trackers; Refs. 4 and 6 provide the same formulation for gimbal star trackers; and Ref. 5 compares the performance between these two star-tracker configurations. The body formulation method (referred to as the Lefferts–Markley–Shuster filter) essentially assumes that the attitude knowledge error is originated from the destination frame,

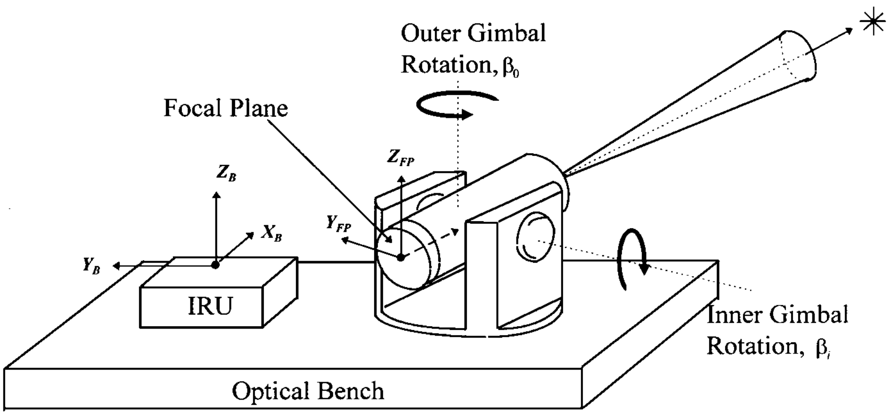

Fig. 1 IRU and optical payload pointing system configuration.

i.e., spacecraft body frame. References 7 and 8 (referred to as the Thompson–Quasius filter) show the Kalman filter derivation using the inertial formulation. In contrast to the body formulation, the inertial formulation method assumes that the attitude knowledge error is originated from the source frame, i.e., inertial frame. One obvious question from these two formulations is this: Do they provide the same solutions?

Reference 12 discussed both the body and inertial formulations and showed that the two formulations are equivalent for a configuration using fixed-head star trackers. This paper provides a complete and alternative derivation of the attitude determination equations for both the spacecraft body and inertial formulations. The derivation takes advantage of the skew-symmetric properties of the attitude kinematics equation. In this paper, this alternative approach is applied to the system that utilizes an optical payload pointing system and body-mounted gyros.

The paper begins with a discussion of the extended Kalman filter equation for attitude determination applications. The state and measurement equation derivations for both the body and inertial formulations are presented, and derivations of the Kalman filter state and measurement equations are shown using the skew-symmetric matrix properties of Poisson's attitude kinematics equation. The paper also shows that both formulations will essentially produce identical results. However, the body formulation is conceptually easier to understand, whereas the inertial formulation reduces the amount of computational time required to arrive at the same results. To show the filter performance using an optical payload pointing system, an attitude determination performance example is shown for an agile low-Earth-orbit satellite system.

Extended Kalman Filter for Spacecraft Attitude Determination Systems

An extended Kalman filter has been used for spacecraft attitude determination systems.[2–8,12] These spacecraft attitude determination systems typically use a six-state Kalman filter as their standard system. The states are three attitude and three gyro bias errors. During a calibration period, e.g., gyro and payload calibration, the filter may include additional states such as gyro scale factor, gyro alignments, inner/outer gimbal alignments, etc. In this paper, we will emphasize the attitude determination period that utilizes the six-state extended Kalman filter mechanization; however, the results can easily be extended to include any additional states, e.g., payload and gyro states. The following paragraphs review the principal equations for the extended Kalman filter to introduce the mathematical symbols for the sections that follow.

The extended Kalman filter formulation is obtained from the assumed system dynamics equation and the system measurement equation. The continuous versions of the assumed models are in the following form[6,13]:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f[x(t), t] + w(t), \qquad E[w(t)\cdot w(t)^T] = Q(t)\cdot\delta(t-\tau) \tag{1}$$

$$y(t) = h[x(t), t] + v(t), \qquad E[v(t)\cdot v(t)^T] = R(t)\cdot\delta(t-\tau) \tag{2}$$

Then the nonlinear propagation equations for the extended Kalman filter are

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = f[\hat{x}(t), t], \qquad \hat{y}(t) = h[\hat{x}(t), t] \tag{3}$$

where the ^ notation refers to an estimate.

The state error vector is defined by

$$\delta x(t) = x(t) - \hat{x}(t) \tag{4}$$

Linearizing Eqs. (1–3) about a nominal $\delta x(t)$, the linearized state error and measurement equations satisfy the following differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x(t) = F(t)\cdot\delta x(t) + w(t) \tag{5a}$$

$$y(t) = H(t)\cdot\delta x(t) + v(t) \tag{6a}$$

where

$$F(t) = \left.\frac{\partial f(x,t)}{\partial x}\right|_{\hat{x}(t)}, \qquad H(t) = \left.\frac{\partial h(x,t)}{\partial x}\right|_{\hat{x}(t)}$$

The discrete version of Eqs. (5a) and (6a) can be represented as follows:

$$\delta x^{k+1} = \Phi^k\cdot\delta x^k + w^k \tag{5b}$$

$$y^k = H^k\cdot\delta x^k + v^k \tag{6b}$$

In the sections that follow, the propagation equations are derived explicitly for an optical payload pointing system.

State Equation for the Body Formulation

Figure 2 depicts the relationship between inertial, true body, and estimated body frames and their coordinate transformations, e.g., direction cosines, for the body frame formulation. In this formulation, the assumption is that the coordinate transformation or attitude knowledge error is coming from the destination frame, i.e., spacecraft body frame. Then the coordinate transformation relationship between frames can be described as

$$T_I^B = T_{\hat{B}}^B\cdot T_I^{\hat{B}} \tag{7}$$

Differentiating Eq. (7) with respect to time,

$$\frac{\mathrm{d}T_I^B}{\mathrm{d}t} = \frac{\mathrm{d}T_{\hat{B}}^B}{\mathrm{d}t}\cdot T_I^{\hat{B}} + T_{\hat{B}}^B\cdot\frac{\mathrm{d}T_I^{\hat{B}}}{\mathrm{d}t} \tag{8}$$

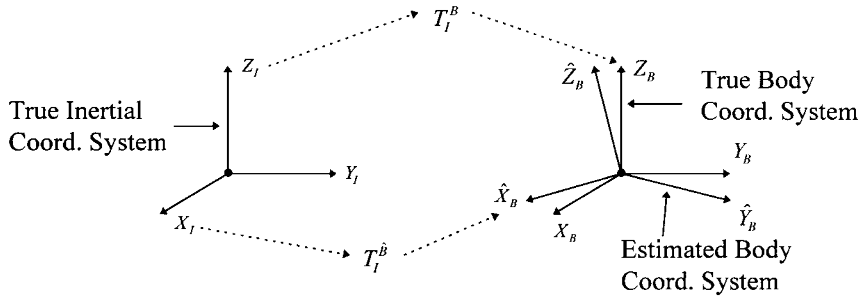

Fig. 2 Coordinate transformation relationship for the body formulation.

From attitude kinematics, we know that two reference frames that evolve with time satisfy the following Poisson's kinematics equation for attitude:

$$\frac{\mathrm{d}T_I^B}{\mathrm{d}t} = -\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \cdot T_I^B = -\mathrm{skew}(\boldsymbol{\Omega}) \cdot T_I^B \tag{9}$$

Rearranging Eq. (9) and substituting Eq. (8), Eq. (9) becomes

$$\mathrm{skew}(\boldsymbol{\Omega}) = T_I^B \cdot \left[\frac{\mathrm{d}T_I^B}{\mathrm{d}t}\right]^T = T_{\hat{B}}^B \cdot T_I^{\hat{B}} \cdot \left[\frac{\mathrm{d}T_{\hat{B}}^B}{\mathrm{d}t} \cdot T_I^{\hat{B}} + T_{\hat{B}}^B \cdot \frac{\mathrm{d}T_I^{\hat{B}}}{\mathrm{d}t}\right]^T \tag{10}$$

$$= T_{\hat{B}}^B \cdot T_I^{\hat{B}} \cdot \left[T_I^{\hat{B}}\right]^T \cdot \left[\frac{\mathrm{d}T_{\hat{B}}^B}{\mathrm{d}t}\right]^T + T_{\hat{B}}^B \cdot T_I^{\hat{B}} \cdot \left[\frac{\mathrm{d}T_I^{\hat{B}}}{\mathrm{d}t}\right]^T \cdot \left[T_{\hat{B}}^B\right]^T \tag{11}$$

Using the coordinate transformation property, $[T_I^{\hat{B}}]^{-1} = [T_I^{\hat{B}}]^T$, and Poisson's kinematics equation for the estimated transformation,

$$\mathrm{skew}(\hat{\boldsymbol{\Omega}}) = T_I^{\hat{B}} \cdot \left[\frac{\mathrm{d}T_I^{\hat{B}}}{\mathrm{d}t}\right]^T \tag{12}$$

Eq. (11) becomes

$$\mathrm{skew}(\boldsymbol{\Omega}) = T_{\hat{B}}^B \cdot \left[\frac{\mathrm{d}T_{\hat{B}}^B}{\mathrm{d}t}\right]^T + T_{\hat{B}}^B \cdot \mathrm{skew}(\hat{\boldsymbol{\Omega}}) \cdot \left[T_{\hat{B}}^B\right]^T \tag{13}$$

The skew-symmetric matrix has the following property:

$$T_{\hat{B}}^B \cdot \mathrm{skew}(\hat{\boldsymbol{\Omega}}) \cdot \left[T_{\hat{B}}^B\right]^T = \mathrm{skew}\left(T_{\hat{B}}^B \cdot \hat{\boldsymbol{\Omega}}\right)$$

Therefore,

$$\mathrm{skew}(\boldsymbol{\Omega}) = T_{\hat{B}}^B \cdot \left[\frac{\mathrm{d}T_{\hat{B}}^B}{\mathrm{d}t}\right]^T + \mathrm{skew}\left(T_{\hat{B}}^B \cdot \hat{\boldsymbol{\Omega}}\right) \tag{14}$$

Assuming a small angle approximation, the first term becomes

$$T_{\hat{B}}^B \cdot \frac{\mathrm{d}T_{\hat{B}}^B}{\mathrm{d}t} = [I - \mathrm{skew}(\hat{\boldsymbol{\alpha}})] \cdot \mathrm{skew}\left(\frac{\mathrm{d}\hat{\boldsymbol{\alpha}}}{\mathrm{d}t}\right) \approx \mathrm{skew}\left(\frac{\mathrm{d}\hat{\boldsymbol{\alpha}}}{\mathrm{d}t}\right) \tag{15}$$

Substituting back to Eq. (14),

$$\mathrm{skew}(\boldsymbol{\Omega}) = \mathrm{skew}\left(\frac{\mathrm{d}\hat{\boldsymbol{\alpha}}}{\mathrm{d}t}\right) + \mathrm{skew}\left(T_{\hat{B}}^B \cdot \hat{\boldsymbol{\Omega}}\right) \tag{16}$$

Because each term inside the parentheses is a vector, then

$$\boldsymbol{\Omega} = \frac{\mathrm{d}\hat{\boldsymbol{\alpha}}}{\mathrm{d}t} + T_{\hat{B}}^B \cdot \hat{\boldsymbol{\Omega}} \tag{17}$$

Rearranging the terms, we obtain the Kalman filter state equation for the body formulation attitude error state:

$$\frac{\mathrm{d}\hat{\boldsymbol{\alpha}}}{\mathrm{d}t} = (\boldsymbol{\Omega} - \hat{\boldsymbol{\Omega}}) + \mathrm{skew}(\hat{\boldsymbol{\alpha}}) \cdot \hat{\boldsymbol{\Omega}} \tag{18}$$

Because the skew-symmetric matrix satisfies the property

$$\mathrm{skew}(\hat{\boldsymbol{\alpha}}) \cdot \hat{\boldsymbol{\Omega}} = -\mathrm{skew}(\hat{\boldsymbol{\Omega}}) \cdot \hat{\boldsymbol{\alpha}}$$

then Eq. (18) can be written as

$$\frac{\mathrm{d}\hat{\boldsymbol{\alpha}}}{\mathrm{d}t} = \hat{\boldsymbol{b}} - \mathrm{skew}(\hat{\boldsymbol{\Omega}}) \cdot \hat{\boldsymbol{\alpha}} \tag{19}$$

The first term is the gyro bias error state for the body formulation. The gyro bias error is the difference between the true rate and estimated rate:

$$\hat{\boldsymbol{b}} = \boldsymbol{\Omega} - \hat{\boldsymbol{\Omega}} \tag{20}$$

In a stable temperature environment, the gyro bias state is typically constant or slowly time varying with temporal correlation in the order of hours. Hence, the gyro bias equation is

$$\frac{\mathrm{d}\hat{\boldsymbol{b}}}{\mathrm{d}t} = 0 \tag{21}$$

Therefore, the Kalman filter state equation for the body formulation can be written as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = F_B \cdot \boldsymbol{x} = \begin{bmatrix} \dfrac{\mathrm{d}\hat{\boldsymbol{\alpha}}}{\mathrm{d}t} \\ \dfrac{\mathrm{d}\hat{\boldsymbol{b}}}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\mathrm{skew}(\hat{\boldsymbol{\Omega}}) & I_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{\alpha}} \\ \hat{\boldsymbol{b}} \end{bmatrix} \tag{22a}$$

For a small spacecraft rate, the discrete version of F_B, Φ_B, can be approximated as

$$\Phi_B^k = \begin{bmatrix} I_{3\times 3} - \mathrm{skew}(\Delta t \cdot \hat{\boldsymbol{\Omega}}^k) & \Delta t \cdot I_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix} \tag{22b}$$

Measurement Equation for the Body Formulation

For an optical payload pointing system as given in Fig. 1, the gimballed control system commands the payload to a star location as given by the star catalog. Once the payload acquires the star in its field of view, the control system maintains inertial pointing until the next available command, and then after a period of focal plane measurement processing, the attitude determination system begins to process the star measurement for the Kalman filter update.

The star measurements from the focal plane are essentially the projection of the unit vector to the star on the plane normal to the payload line of sight. These measurements can be described as the transformation of the star vector from the inertial to the payload focal plane coordinates as follows:

$$\boldsymbol{Y}_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot T_B^{\mathrm{FP}} \cdot T_I^B \cdot \boldsymbol{R}_I^{\mathrm{Star}} \tag{23}$$

Including the estimated and error transformations as given by Fig. 2 and assuming that the focal plane and body coordinates have been calibrated, Eq. (23) becomes

$$\boldsymbol{Y}_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot T_B^{\mathrm{FP}} \cdot T_{\hat{B}}^B \cdot T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{Star}} \tag{24}$$

For an attitude determination system using an optical payload pointing system, the body to focal plane coordinate transformation is a

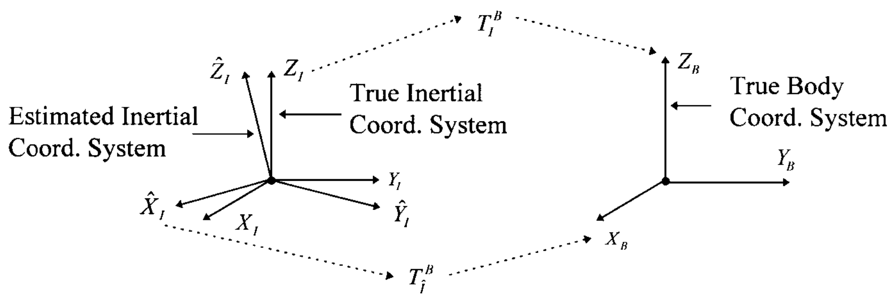

Fig. 3 Coordinate transformation relationship for the inertial formulation.

function of the outer and inner gimballed angles. For the configuration as given in Fig. 1, the transformation is (azimuth rotation about Z_B axis followed by elevation rotation about Y_B axis)

$$T_B^{\mathrm{FP}} = \begin{bmatrix} \cos(\beta_i)\cdot\cos(\beta_o) & \cos(\beta_i)\cdot\sin(\beta_o) & -\sin(\beta_i) \\ -\sin(\beta_o) & \cos(\beta_o) & 0 \\ \sin(\beta_i)\cdot\cos(\beta_o) & \sin(\beta_i)\cdot\sin(\beta_o) & \cos(\beta_i) \end{bmatrix} \tag{25}$$

The expected output of the star on the focal plane can be expressed using the estimated coordinate transformation and the star vector from the star catalog as follows:

$$\boldsymbol{Y}_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot T_{\hat{B}}^{\mathrm{FP}} \cdot T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{StarCat}} \tag{26}$$

Then the residual for the Kalman filter state correction is

$$\boldsymbol{res} = \boldsymbol{Y}_M - \boldsymbol{Y}_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \left(T_B^{\mathrm{FP}} \cdot T_{\hat{B}}^{B} \cdot T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{Star}} - T_{\hat{B}}^{\mathrm{FP}} \cdot T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{StarCat}}\right) \tag{27}$$

Assuming a small angle approximation, the partial derivative for the measurement matrix H is

$$\frac{\partial \boldsymbol{res}}{\partial \hat{\boldsymbol{\alpha}}} = \frac{\partial}{\partial \hat{\boldsymbol{\alpha}}}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \left\{T_B^{\mathrm{FP}} \cdot [I_{3\times 3} - \mathrm{skew}(\hat{\boldsymbol{\alpha}})] \cdot T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{Star}} - T_{\hat{B}}^{\mathrm{FP}} \cdot T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{StarCat}}\right\}\right) \tag{28}$$

Eliminating the terms that are not a function of $\hat{\boldsymbol{\alpha}}$,

$$\frac{\partial \boldsymbol{res}}{\partial \hat{\boldsymbol{\alpha}}} = \frac{\partial}{\partial \hat{\boldsymbol{\alpha}}}\left\{-\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot T_B^{\mathrm{FP}} \cdot \mathrm{skew}(\hat{\boldsymbol{\alpha}}) \cdot T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{Star}}\right\} \tag{29}$$

Using the skew-symmetric matrix property, $\mathrm{skew}(\hat{\boldsymbol{\alpha}}) \cdot T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{Star}} = -\mathrm{skew}(T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{Star}}) \cdot \hat{\boldsymbol{\alpha}}$ (Ref. 14), Eq. (29) becomes

$$\frac{\partial \boldsymbol{res}}{\partial \hat{\boldsymbol{\alpha}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot T_B^{\mathrm{FP}} \cdot \mathrm{skew}\left(T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{Star}}\right) \tag{30}$$

The partial derivative of $\boldsymbol{res}$ with respect to the gyro bias states is zero. They are not observable from the payload measurement. However, they will be coupled through the state transition matrix over the time period between updates. Therefore, the Kalman filter measurement matrix H for the body formulation is

$$H_B = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot T_B^{\mathrm{FP}} \cdot \mathrm{skew}\left(T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{Star}}\right) \quad 0_{2\times 3}\right] \tag{31}$$

For the optical payload pointing configuration as shown in Fig. 1, the measurement matrix is

State Equation for the Inertial Formulation

For the inertial formulation case, the assumption is that the attitude knowledge error comes from the source frame or inertial frame. Figure 3 shows the relationship between the true body, true inertial, and estimated inertial coordinate frames. The relationship between coordinate frames can be expressed as follows:

$$T_I^B = T_{\hat{I}}^B \cdot T_I^{\hat{I}} \tag{33}$$

In contrast to the body formulation, the estimated inertial to body transformation is postmultiplied by the transformation matrix error $T_I^{\hat{I}}$ to get to the true inertial to body transformation. Differentiating Eq. (33),

$$\frac{\mathrm{d}T_I^B}{\mathrm{d}t} = \frac{\mathrm{d}T_{\hat{I}}^B}{\mathrm{d}t} \cdot T_I^{\hat{I}} + T_{\hat{I}}^B \cdot \frac{\mathrm{d}T_I^{\hat{I}}}{\mathrm{d}t} \tag{34}$$

Using Eqs. (9), (33), and (34), we get

$$\mathrm{skew}(\boldsymbol{\Omega}) = T_I^B \cdot \left[\frac{\mathrm{d}T_I^B}{\mathrm{d}t}\right]^T = T_{\hat{I}}^B \cdot T_I^{\hat{I}} \cdot \left[\frac{\mathrm{d}T_{\hat{I}}^B}{\mathrm{d}t} \cdot T_I^{\hat{I}} + T_{\hat{I}}^B \cdot \frac{\mathrm{d}T_I^{\hat{I}}}{\mathrm{d}t}\right]^T \tag{35}$$

Using the direction cosine property and Poisson's kinematics equation,

$$\mathrm{skew}(\tilde{\boldsymbol{\Omega}}) = T_I^{\hat{I}} \cdot \frac{\mathrm{d}T_I^{\hat{I}}}{\mathrm{d}t} \tag{36}$$

it yields

$$\mathrm{skew}(\boldsymbol{\Omega}) = T_{\hat{I}}^B \cdot \left[\frac{\mathrm{d}T_{\hat{I}}^B}{\mathrm{d}t}\right]^T + T_{\hat{I}}^B \cdot \mathrm{skew}(\tilde{\boldsymbol{\Omega}}) \cdot \left[T_{\hat{I}}^B\right]^T \tag{37}$$

The skew-symmetric matrix satisfies the following property:

$$T_{\hat{I}}^B \cdot \mathrm{skew}(\tilde{\boldsymbol{\Omega}}) \cdot \left[T_{\hat{I}}^B\right]^T = \mathrm{skew}\left(T_{\hat{I}}^B \cdot \tilde{\boldsymbol{\Omega}}\right) \tag{38}$$

Using this property and Poisson's kinematics equation, Eq. (37) can be written as

$$\mathrm{skew}(\boldsymbol{\Omega}) = \mathrm{skew}(\hat{\boldsymbol{\Omega}}) + \mathrm{skew}\left(T_{\hat{I}}^B \cdot \tilde{\boldsymbol{\Omega}}\right) \tag{39}$$

or in vector form it becomes

$$\boldsymbol{\Omega} = \hat{\boldsymbol{\Omega}} + T_{\hat{I}}^B \cdot \tilde{\boldsymbol{\Omega}} \tag{40}$$

Assuming a small angle approximation, the state equation for the inertial formulation attitude error is

$$\tilde{\boldsymbol{\Omega}} \approx \frac{\mathrm{d}\hat{\boldsymbol{A}}}{\mathrm{d}t} \approx \left(T_{\hat{I}}^B\right) \cdot (\boldsymbol{\Omega} - \hat{\boldsymbol{\Omega}}) \tag{41}$$

The gyro bias error state for the inertial formulation is

$$\hat{\boldsymbol{B}} = \boldsymbol{\Omega} - \hat{\boldsymbol{\Omega}} \tag{42}$$

$$H_B = \left\{\begin{bmatrix} \cos(\beta_i)\cdot\cos(\beta_o) & \cos(\beta_i)\cdot\sin(\beta_o) & -\sin(\beta_i) \\ -\sin(\beta_o) & \cos(\beta_o) & 0 \end{bmatrix} \cdot \mathrm{skew}\left(T_I^{\hat{B}} \cdot \boldsymbol{R}_I^{\mathrm{Star}}\right) \quad 0_{2\times 3}\right\} \tag{32}$$

For a constant gyro bias, the gyro bias differential equation is

$$\frac{\mathrm{d}\hat{\boldsymbol{B}}}{\mathrm{d}t} = 0 \tag{43}$$

Therefore, the Kalman filter state equation for the inertial formulation is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F_I \cdot x = \begin{bmatrix} \dfrac{\mathrm{d}\hat{\boldsymbol{A}}}{\mathrm{d}t} \\ \dfrac{\mathrm{d}\hat{\boldsymbol{B}}}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} 0_{3\times 3} & \left(T_{\hat{I}}^{B}\right)^T \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{A}} \\ \hat{\boldsymbol{B}} \end{bmatrix} \tag{44a}$$

The discrete version of F_I, Φ_I, is

$$\Phi_I^k = \begin{Bmatrix} I_{3\times 3} & \displaystyle\int_0^{\Delta t} \left[T_{\hat{I}}^{B}(\tau)\right]^T \cdot \mathrm{d}\tau \\ 0_{3\times 3} & I_{3\times 3} \end{Bmatrix} \tag{44b}$$

Compared with the body formulation in Eqs. (44a) and (44b), the preceding state equations are considered simpler.

Measurement Equation for the Inertial Formulation

For the inertial formulation case, the star measurement equation is

$$\boldsymbol{Y}_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot T_B^{\mathrm{FP}} \cdot T_{\hat{I}}^{B} \cdot T_I^{\hat{I}} \cdot \boldsymbol{R}_I^{\mathrm{Star}} \tag{45}$$

Then the residual for the Kalman filter states correction becomes

$$\boldsymbol{res} = \boldsymbol{Y}_M - \boldsymbol{Y}_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \left(T_B^{\mathrm{FP}} \cdot T_{\hat{I}}^{B} \cdot T_I^{\hat{I}} \cdot \boldsymbol{R}_I^{\mathrm{Star}} - T_B^{\mathrm{FP}} \cdot T_{\hat{I}}^{B} \cdot \boldsymbol{R}_I^{\mathrm{StarCat}}\right) \tag{46}$$

Assuming a small angle approximation, the partial derivative for the measurement matrix H is

$$\frac{\partial \boldsymbol{res}}{\partial \hat{\boldsymbol{A}}} = \frac{\partial}{\partial \hat{\boldsymbol{A}}}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \left\{T_B^{\mathrm{FP}} \cdot T_{\hat{I}}^{B} \cdot \left[I_{3\times 3} - \mathrm{skew}(\hat{\boldsymbol{A}})\right] \cdot \boldsymbol{R}_I^{\mathrm{Star}} - T_B^{\mathrm{FP}} \cdot T_{\hat{I}}^{B} \cdot \boldsymbol{R}_I^{\mathrm{StarCat}}\right\}\right) \tag{47}$$

$$= \frac{\partial}{\partial \hat{\boldsymbol{A}}}\left\{-\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot T_B^{\mathrm{FP}} \cdot T_{\hat{I}}^{B} \cdot \mathrm{skew}(\hat{\boldsymbol{A}}) \cdot \boldsymbol{R}_I^{\mathrm{Star}}\right\} \tag{48}$$

Using the skew-symmetric matrix property, Eq. (48) becomes

$$\frac{\partial \boldsymbol{res}}{\partial \hat{\boldsymbol{A}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot T_B^{\mathrm{FP}} \cdot T_{\hat{I}}^{B} \cdot \mathrm{skew}\left(\boldsymbol{R}_I^{\mathrm{Star}}\right) \tag{49}$$

Similar to the body formulation case, the partial derivative of $\boldsymbol{res}$ with respect to the gyro bias states is zero. Therefore, the measurement matrix H for the inertial formulation is

$$H_I = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot T_B^{\mathrm{FP}} \cdot T_{\hat{I}}^{B} \cdot \mathrm{skew}\left(\boldsymbol{R}_I^{\mathrm{Star}}\right) \quad 0_{2\times 3}\right] \tag{50}$$

For the configuration as shown in Fig. 1, the matrix becomes

$$H_I = \left\{\begin{bmatrix} \cos(\beta_i)\cdot\cos(\beta_o) & \cos(\beta_i)\cdot\sin(\beta_o) & -\sin(\beta_i) \\ -\sin(\beta_o) & \cos(\beta_o) & 0 \end{bmatrix} \cdot T_{\hat{I}}^{B} \cdot \mathrm{skew}\left(\boldsymbol{R}_I^{\mathrm{Star}}\right) \quad 0_{2\times 3}\right\} \tag{51}$$

Relationship Between Body and Inertial Formulations

We have derived the propagation equations for both body and inertial formulations. Two obvious questions are these: Do they produce the same results, and what is the relationship between the two formulations? In this section, we are going to answer these questions.

Using the true transformation from inertial to body coordinate in Eqs. (7) and (33), the two formulations satisfy the following relationship:

$$T_{\hat{B}}^{B} \cdot T_I^{\hat{B}} = T_{\hat{I}}^{B} \cdot T_I^{\hat{I}} \tag{52}$$

Assuming small angle approximations, the equation becomes

$$[I_{3\times 3} - \mathrm{skew}(\hat{\boldsymbol{\alpha}})] \cdot T_I^{\hat{B}} = T_{\hat{I}}^{B} \cdot [I_{3\times 3} - \mathrm{skew}(\hat{\boldsymbol{A}})] \tag{53}$$

By definition, the estimated transformations from inertial to body are equal, i.e., $T_I^{\hat{B}} = T_{\hat{I}}^{B}$. Then,

$$\mathrm{skew}(\hat{\boldsymbol{\alpha}}) \cdot T_I^{\hat{B}} = T_I^{\hat{B}} \cdot \mathrm{skew}(\hat{\boldsymbol{A}}) \tag{54}$$

Rearranging the terms,

$$\left(T_I^{\hat{B}}\right)^T \cdot \mathrm{skew}(\hat{\boldsymbol{\alpha}}) \cdot T_I^{\hat{B}} = \mathrm{skew}(\hat{\boldsymbol{A}}) \tag{55}$$

Using the skew-symmetric matrix property, $(T_I^{\hat{B}})^T \cdot \mathrm{skew}(\hat{\boldsymbol{\alpha}}) \cdot T_I^{\hat{B}} = \mathrm{skew}(T_I^{\hat{B}} \cdot \hat{\boldsymbol{\alpha}})$, then the attitude error relationship between the body and inertial formulations is

$$T_I^{\hat{B}} \cdot \hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{A}} \tag{56}$$

As a result, the measurement and state equations for both formulations have the following relationship:

$$H_I \cdot \begin{bmatrix} T_I^{\hat{B}} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix} = H_B \tag{57}$$

$$\begin{bmatrix} T_I^{\hat{B}} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix}^{-1} \cdot F_I \cdot \begin{bmatrix} T_I^{\hat{B}} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix} = F_B \tag{58}$$

Similar relationships exist for the Kalman filter covariance and gain matrices. In conclusion, with a single transformation from inertial to body, the inertial formulation attitude estimate is identical to the body formulation attitude estimate.

The formulation in inertial as opposed to body coordinates has some advantages in the filter implementation. Practically, the state transition matrix of the inertial formulation is easy to compute, i.e., block triangular form. As a result, the numerical load of the covariance propagation between star sightings is less intensive as compared with the body formulation method.

Example

In this section, an application of the attitude determination system using an optical payload pointing system is shown. The attitude estimator equation selected is the discrete Kalman filter version of the body formulation. The example shown is an arbitrary system of a low-Earth-orbit spacecraft that carries an optical payload pointing system as shown in Fig. 1. The spacecraft operates in a circular orbit with an altitude of 750 km and an inclination of 110 deg. To keep the sunlight out of the $+X_B$ axis side, the satellite is yaw steered with typical roll, pitch, and yaw rate profiles as shown in Fig. 4. The yaw steering rate is commanded along the $+Z_B$ axis, and these spacecraft rate profiles depend on the location of the sun relative to the spacecraft body frame.

For this yaw-steered spacecraft example, the proposed attitude determination system using the payload pointing system provides an

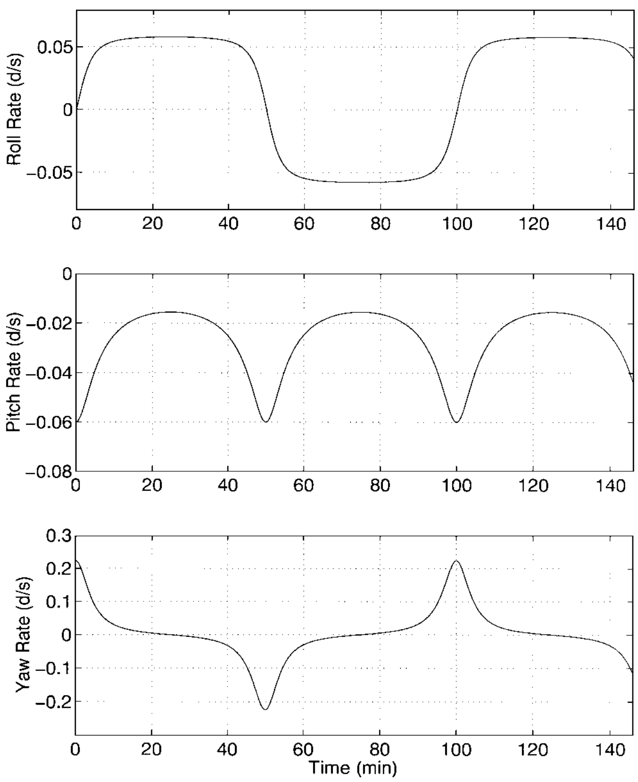

Fig. 4 Spacecraft attitude rate profile.

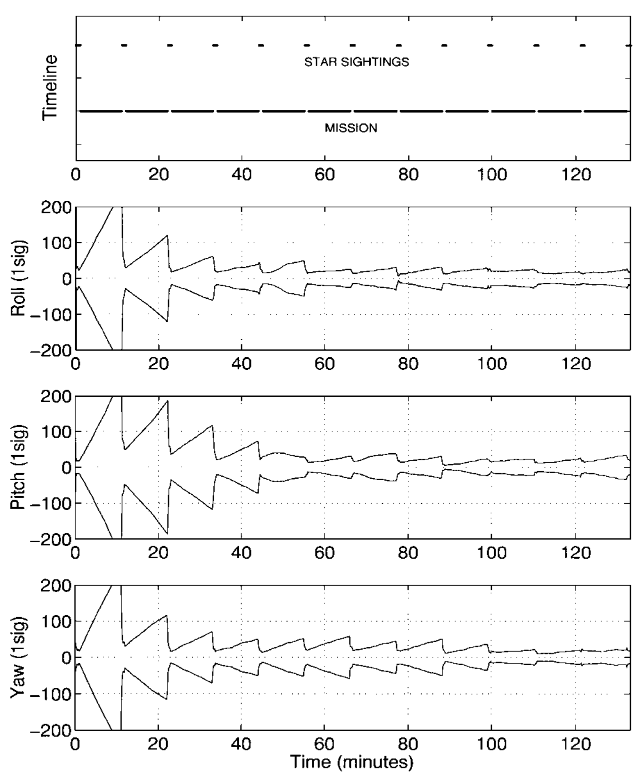

Fig. 5 Spacecraft attitude error estimate in microradians (body formulation).

additional advantage. For the traditional configuration using fixed-head star trackers, large spacecraft rate maneuver, e.g., yaw steering, during star measurements degrades the attitude determination performance due to the image motion of the star on the focal plane. In contrast, the proposed configuration will point the payload inertially to the star; hence the star image on the focal plane is relatively motionless, and the star processing algorithm can process the image more accurately as time evolves.

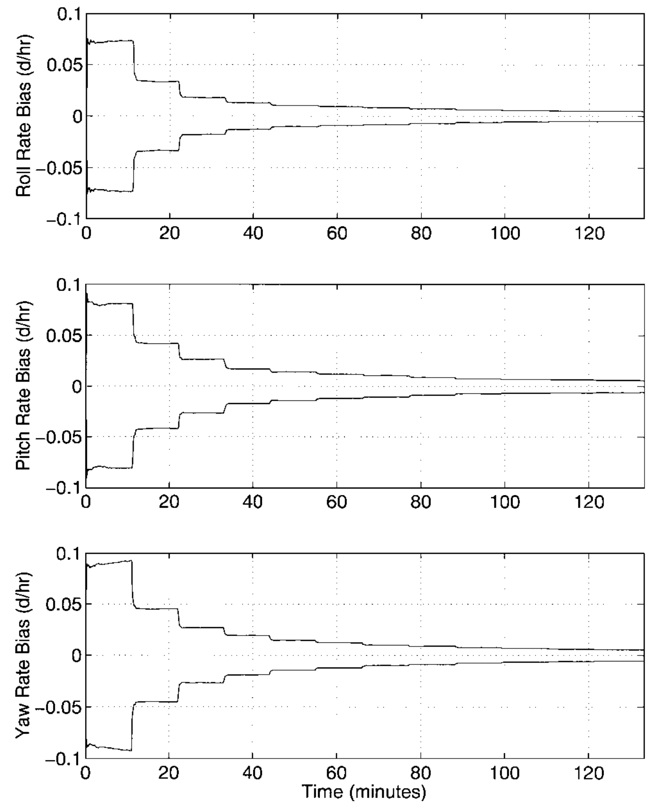

Fig. 6 Gyro bias error estimate.

The optical pointing system selected is a high-bandwidth and high-accuracy system. The bandwidth selected is such that the payload can slew 180-deg angular excursion to within 4 s. The accuracy of the frame-to-frame measurement is 100-μrad (one-sigma) Gaussian white noise, and the sample rate for this frame-to-frame measurement is 330 ms. This error is relatively large; however, measurements with a multiple frame averaging process reduce this random error by the square root of the number of available frames. For the Kalman filter design, this multiple frame averaging measurement error is used to define the Kalman filter measurement covariance matrix R.

The selected Inertial Reference Unit (IRU) for this example is a ring laser gyro. The gyro model used is[3]

$$\omega_{\text{gyro}} = (1 + SF) \cdot \omega_{\text{true}} + b_{\text{gyro}} + \eta_r \tag{59}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} b_{\text{gyro}} = \eta_a \tag{60}$$

The performance parameters for the gyro are as follows: The scale factor residual is 1.0 part per million, the initial gyro bias is 0.1 deg/h, the angle random walk is 0.001 deg/h$^{0.5}$, and the rate random walk is $50e$-6 deg/h/h$^{0.5}$. The angle and rate random walk values are selected to define the Kalman filter state covariance matrix Q.

In this example, the payload pointing system performs star sightings for a 60-s period with an update period at every 10 min. These periods have been selected arbitrarily for this example and should be selected accordingly based on the mission and attitude knowledge requirements. During the 10-min update period, the payload supports the mission, and the Kalman filter propagates the attitude estimate states using the rate measurements from the gyros. The sample rate for attitude propagation and star measurements is 0.1 Hz. This sample rate translates to six star sightings per 60-s measurement period. The 10-s period for each star measurement includes the payload slewing and settling with a maximum period of 4 s and focal plane image processing with a minimum period of 6 s.

Using the performance parameters as described earlier, a Monte Carlo simulation with 50 random seeds has been done. Typical results of this simulation are shown in Figs. 5 and 6 for the one-sigma envelopes of the body formulation attitude error and gyro bias error states. Initial attitude errors of 0.1 deg in each axis were used. The attitude errors converge immediately to the accuracy of the payload after the first update. However, the Kalman filter needs two updates before the gyro rate bias starts to converge. Thereafter, typical steady-state performance is indicated. The attitude errors show a sawtooth pattern, and the gyro bias errors converge to negligible constant values.

The results suggest that a spacecraft attitude determination system using an optical payload pointing system is possible as long as star sightings for measurement updates are provided periodically. The simulation results also show that the attitude estimate has adequate performance once the gyro bias state reaches the steady-state condition.

Conclusion

This paper has proposed the use of an optical payload pointing system for supporting the main payload mission and precision attitude determination system. This proposed configuration eliminates the need for star trackers or other inertial reference systems, thereby reducing the spacecraft development cost and total system weight. Derivations of the Kalman filter state and measurement equations for both the spacecraft body and inertial formulations have been shown using the attitude kinematics equation. The paper shows that both formulations provide the same results. However, the body formulation is conceptually easier to understand, and the inertial formulation needs less processing time. The results of an attitude determination system example for an agile low-Earth-orbit satellite show that the proposed configuration provides adequate attitude estimate information as long as star measurements are available periodically and gyro bias errors have reached steady state.

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